Quasioptics with mode conversion: theory and applications

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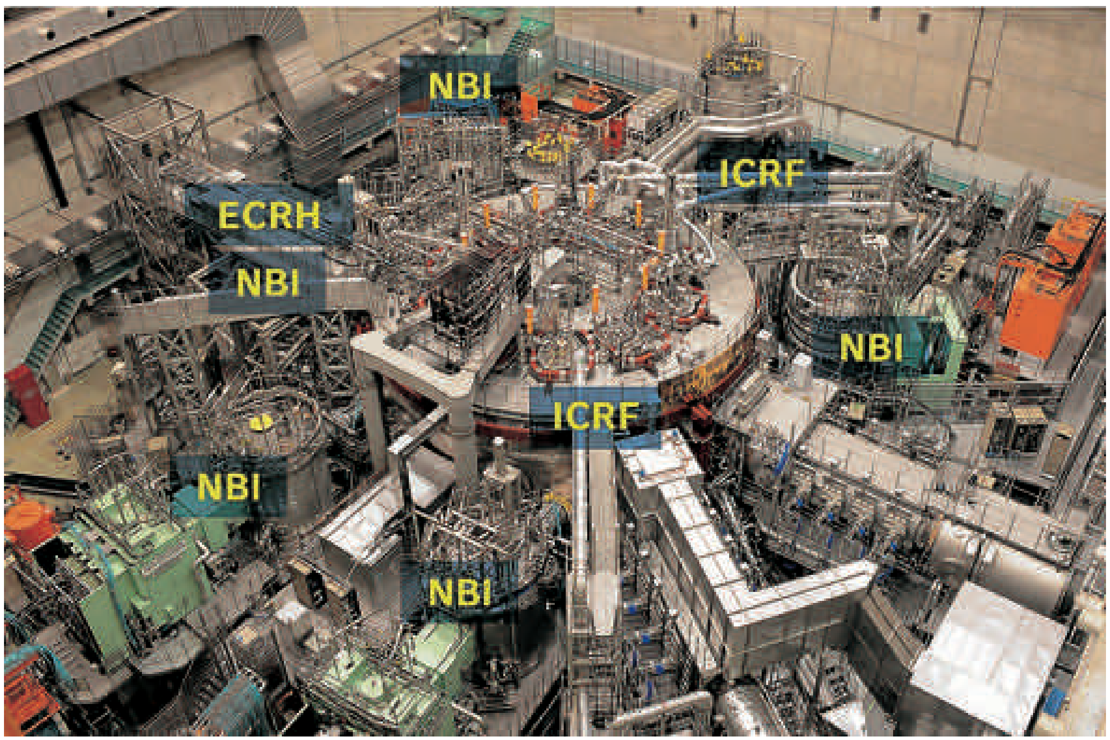
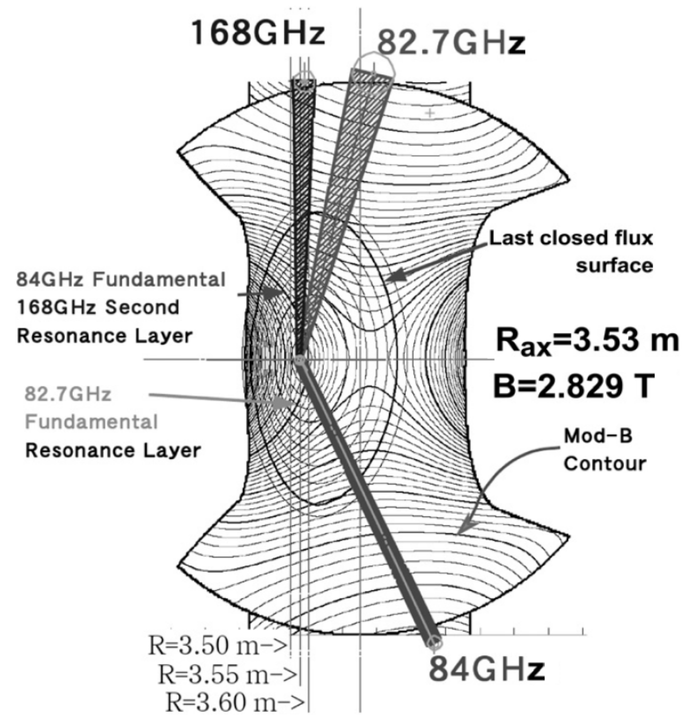
Collaboration with NIFS on ECH modeling at LHD: directions

- **Problem 1:** Adequate modeling of the wave dissipation in the focal region
  - The currently adopted GO model predicts unphysical infinite power absorption.
  - Beam tracing is not an option, the beam profile must be actually calculated.
  - The existing general quasioptical theories are controversial.





The work that motivated the LHD team to look beyond beam tracing

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ECRH power deposition from a quasi-optical point of view

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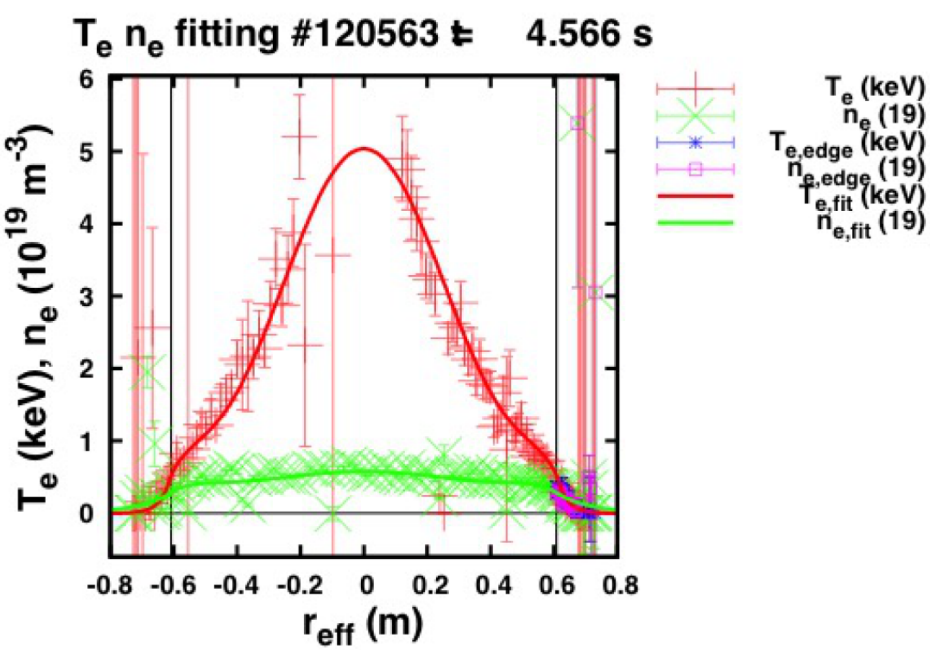
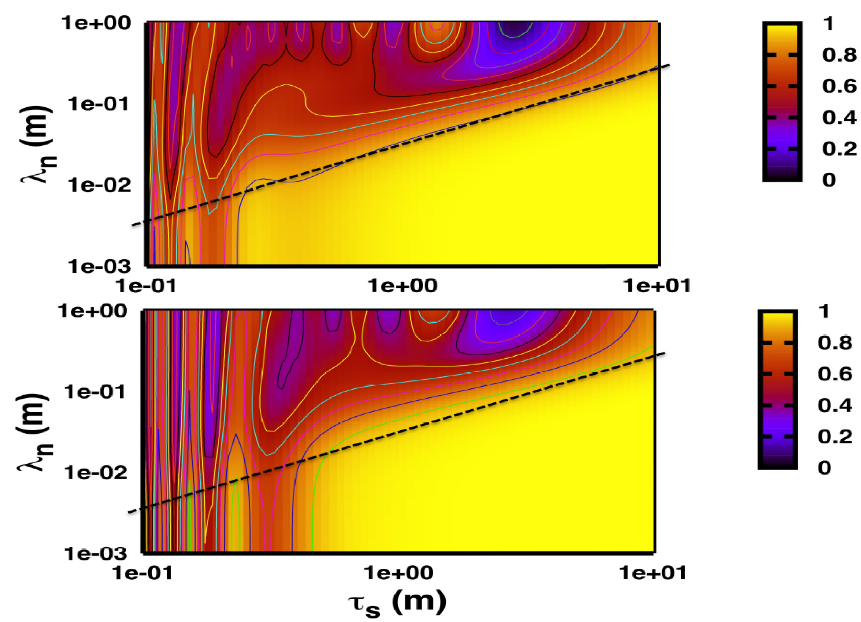
Abstract

A quasi-optical description of the propagation and damping of the slowly varying wave amplitude across an arbitrary electron cyclotron wave beam is presented. This model goes well beyond those implemented in existing beam tracing codes, which typically require the spatial inhomogeneity across the wave beam to be small. The present model allows an accurate description of the wave beam evolution in the region of electron cyclotron power deposition, where the latter condition is quite generally broken. The additional physical effects from spatial inhomogeneity and dispersion included in the quasi-optical model are discussed in relation to their consequences for the power deposition profile. Quite generally, a broader power deposition profile is obtained in the quasi-optical calculations. The importance of these effects is analysed in a number of scans varying the injection geometry for typical conditions in both the ITER and the TEXTOR tokamak. Optimization of the power deposition profile towards a minimal width is found to require a focused wave beam with a waist of typically 2 cm width localized near the electron cyclotron resonance region. Calculations are also presented for beams injected from the ITER Upper Port electron cyclotron resonance heating (ECRH) launcher as it is currently being designed. These show that the additional power deposition profile broadening from quasi-optical effects may result in a drop in the predicted efficiency for neoclassical tearing mode or sawtooth control by up to a factor of 2.

Collaboration with NIFS on ECH modeling at LHD: directions

- **Problem 2:** accounting for the O-X mode conversion at the plasma edge
  - A single-mode (O-mode) operation is desired in the plasma core.
  - A strong magnetic shear couples the O mode to the X mode at the edge.
  - Need to find conditions at which the wave reaches the core as a pure O mode.
  - So far, only 1D full-wave simulations have been available.





Kubo *et al*, AIP Conf. Proc. 1689, 090006 (2015); Tsujimura et al, Nucl. Fusion 55, 129019 (2015); Notake *et al*, PPCF 47, 531 (2005)...

Outline

NIFS's rationale for the collaboration: the mentioned problems might be solvable using **extended geometrical optics (XGO)** that we recently developed.[†]

- Introduction
  - GO and XGO for nondissipative waves
  - A new ray-tracing code with adiabatic XGO corrections
- New results:

0. A general 2nd-order reduction for dispersion operators of coupled waves
1. A quasioptical equation for beams with mode conversion (being coded at NIFS)
2. An analytical theory of the O-X mode conversion in edge plasma

   Dodin, Ruiz, and Kubo, arXiv:1709.02841

[†] D. E. Ruiz, *Geometric theory of waves and its applications to plasma physics*, PhD thesis (Princeton University, 2017).

Preliminaries: a variational approach to ray tracing

- Assuming Hermitian $\hat{\varepsilon}$, the wave equation has a variational form $\boldsymbol{\delta S = 0}$. The action S can be represented as an asymptotic power expansion in the GO parameter.

$$\hat{\mathbf{D}}\mathbf{E} = 0, \quad \hat{\mathbf{D}} = c^2\hat{\omega}^{-2}\left[\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbb{1}(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}})\right] + \hat{\varepsilon}(t,\mathbf{x},\hat{\omega},\hat{\mathbf{k}}), \quad \hat{\omega} = i\partial_t, \quad \hat{\mathbf{k}} = -i\nabla$$

$$S = \frac{1}{8\pi}\int \mathbf{E}\hat{\mathbf{D}}\mathbf{E}\,dt\,d^3x = S_0 + \epsilon S_1 + \epsilon^2 S_2 + \ldots, \qquad \epsilon = \lambda/L$$

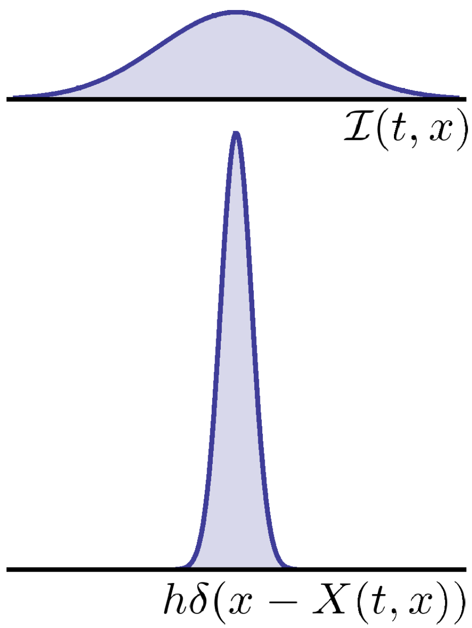
- The zeroth-order truncation gives geometrical optics:

$$S[\theta,\mathcal{I}] = -\int[\partial_t\theta + \omega_0(t,\mathbf{x},\nabla\theta)]\,\mathcal{I}\,dt\,d^3x$$

- "Point-particle" limit: $\mathcal{I} = \boldsymbol{h}\delta(\mathbf{x}-\mathbf{X}(t))$, $\mathbf{K} = \nabla\theta(t,\mathbf{X}(t))$

$$\boxed{S[\mathbf{X},\mathbf{K}] = \int[(\boldsymbol{h}\mathbf{K})\cdot\dot{\mathbf{X}} - \boldsymbol{h}\omega_0(t,\mathbf{X},\mathbf{K})]\,dt}$$

$$\dot{\mathbf{X}} = \partial_{\mathbf{K}}\omega_0(t,\mathbf{X},\mathbf{K}), \quad \dot{\mathbf{K}} = -\partial_{\mathbf{X}}\omega_0(t,\mathbf{X},\mathbf{K})$$



The next-order theory is "extended geometrical optics" (XGO).

- The first-order corrections are *not* diffraction but corrections to GO:

$$S = -\int \big[\partial_t\theta + \omega(t,\mathbf{x},\nabla\theta) - i\mathbf{a}^\dagger(\partial_t + \mathbf{V}\cdot\nabla)\mathbf{a} - \mathbf{a}^\dagger \mathsf{U}(t,\mathbf{x},\nabla\theta)\mathbf{a}\big]\mathcal{I}\, dt\, d^3x$$

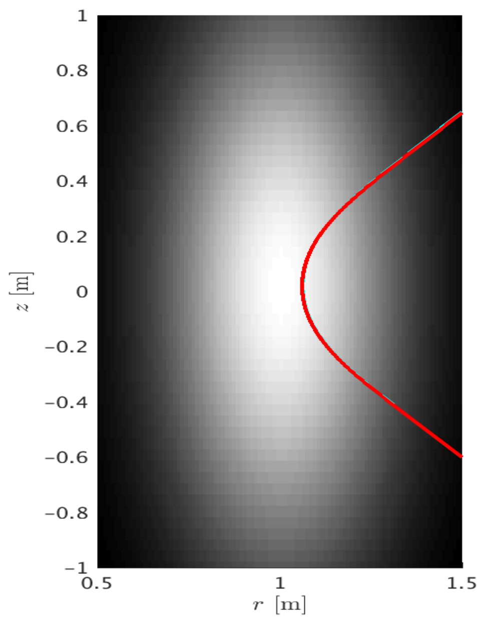
- The point-particle limit gives modified ray equations and amplitude equations:

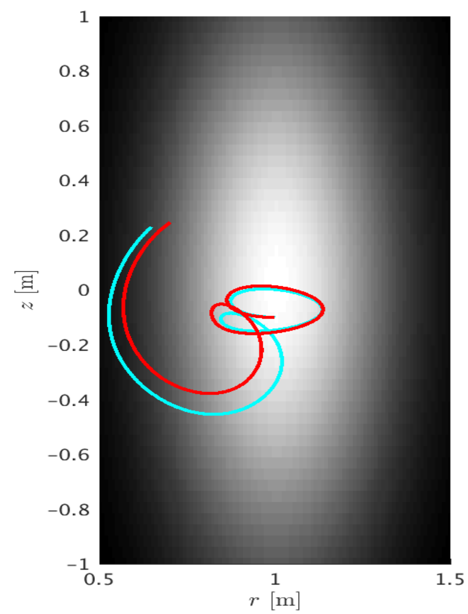
$$\boxed{S[\mathbf{X},\mathbf{K},\mathsf{Z},\mathsf{Z}^\dagger] \approx \boldsymbol{h}\int\big[\mathbf{K}\cdot\dot{\mathbf{X}} - \omega(t,\mathbf{X},\mathbf{K}) + \tfrac{i}{2}(\mathsf{Z}^\dagger\dot{\mathsf{Z}} - \dot{\mathsf{Z}}^\dagger\mathsf{Z}) + \mathsf{Z}^\dagger\mathsf{U}(t,\mathbf{X},\mathbf{K})\mathsf{Z}\big]\,dt}$$

$$\delta\mathbf{X}: \quad \dot{\mathbf{X}} = \partial_{\mathbf{K}}\omega \underbrace{- \mathsf{Z}^\dagger(\partial_{\mathbf{K}}\mathsf{U})\mathsf{Z}}_{O(\epsilon)\ \text{correction}}$$

$$\delta\mathbf{K}: \quad \dot{\mathbf{K}} = -\partial_{\mathbf{X}}\omega \underbrace{+ \mathsf{Z}^\dagger(\partial_{\mathbf{X}}\mathsf{U})\mathsf{Z}}_{\text{“Stern-Gerlach”}},$$

$$\delta\mathsf{Z}^\dagger: \quad \underbrace{i\dot{\mathsf{Z}} = -\mathsf{U}\mathsf{Z}}_{\text{mode conversion}}$$





Ruiz, PhD thesis (2017); Ruiz and Dodin, Phys. Plasmas 24, 055704 (2017); Phys. Rev. A 92, 043805 (2015); Phys. Lett. A 379, 2337 (2015).

Ray-tracing code with adiabatic XGO corrections

- Simulations were performed using a MATLAB-based ray-tracing code that replaces the earlier prototype by Daniel Ruiz. The code is based on the following theory:

  - In GO, the zero eigenvalue of the local dispersion matrix serves as a Hamiltonian:

$$\hat{\mathbf{D}}\boldsymbol{\psi} = 0 \quad \Rightarrow \quad \det \mathbf{D}(X^\mu, K_\mu) = 0 \quad \Rightarrow \quad D_0(X^\mu, K_\mu) = 0$$

$$\det \mathbf{D}(X^\mu, K_\mu) = \prod_q D_q(X^\mu, K_\mu) \equiv \underbrace{D_0(X^\mu, K_\mu)}_{\text{zero eigenvalue}} \underbrace{C(X^\mu, K_\mu)}_{\text{nonzero factor}}$$

  - In adiabatic XGO ($\mathsf{Z} = \mathrm{const}$), $\mathsf{Z}^\dagger \mathsf{U} \mathsf{Z}$ serves as a correction to this Hamiltonian.

$$D = D_0 - \frac{\partial D_0}{\partial K_\alpha}\left(\boldsymbol{\Xi}^\dagger \frac{\partial \boldsymbol{\Xi}}{\partial X^\alpha}\right)_A + \frac{\partial D_0}{\partial X^\alpha}\left(\boldsymbol{\Xi}^\dagger \frac{\partial \boldsymbol{\Xi}}{\partial K_\alpha}\right)_A + \left[\left(\frac{\partial \boldsymbol{\Xi}^\dagger}{\partial K_\alpha}\right)(\mathbf{D} - D_0 \mathbb{1})\left(\frac{\partial \boldsymbol{\Xi}}{\partial X^\alpha}\right)\right]_A$$

$$\boxed{\frac{dX^\mu}{d\tau} = \frac{\partial D(X^\mu, K_\mu)}{\partial K_\mu}, \qquad \frac{dK_\mu}{d\tau} = -\frac{\partial D(X^\mu, K_\mu)}{\partial X^\mu}}$$

Ruiz and Dodin, Phys. Plasmas 24, 055704 (2017).

Two alternative representations of the dispersion matrix

- The new code accounts for ions, has a modular structure, six solvers for benchmarking, and two representations of $\mathbf{D}$ in laboratory coordinates:

  - Assuming the wave equation in the form $\hat{\mathbf{D}}\mathbf{E} = 0$, one can use

$$\boxed{\mathbf{D} = \frac{c^2}{\omega^2}(\mathbf{kk} - k^2\mathbb{1}) + \mathbb{1} + \sum_s \left[-\frac{\omega_{p,s}^2}{\omega^2}\mathbb{1} + \frac{\omega_{p,s}^2(\boldsymbol{\alpha}\cdot\boldsymbol{\Omega}_s)}{\omega(\omega^2 - \Omega_s^2)} - \frac{\omega_{p,s}^2}{\omega^2}\frac{(\boldsymbol{\alpha}\cdot\boldsymbol{\Omega}_s)^2}{\omega^2 - \Omega_s^2} \right]}$$

$$\alpha_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \alpha_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \alpha_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

  - Alternatively, one can use $(i\partial_t - \hat{\boldsymbol{\mathcal{H}}})\boldsymbol{\psi} = 0$, where $\boldsymbol{\psi} = (\mathbf{v}_1\sqrt{4\pi n_1 m_1}, \ldots; \mathbf{E}, \mathbf{B})^{\mathrm{T}}$.

$$\partial_t \mathbf{v}_s = (e_s/m_s)\mathbf{E} + \mathbf{v}_s \times \boldsymbol{\Omega}_s/c, \qquad \partial_t\mathbf{E} = -4\pi e n_{0,s}\mathbf{v}_s + c\nabla\times\mathbf{B}, \qquad \partial_t\mathbf{B} = -c\nabla\times\mathbf{E}$$

$$\boxed{\boldsymbol{\mathcal{H}} = \begin{pmatrix} \omega + \boldsymbol{\alpha}\cdot\boldsymbol{\Omega}_1(\mathbf{x}) & 0 & \ldots & 0 & -i\omega_{p,1}(\mathbf{x}) & 0 \\ 0 & \omega + \boldsymbol{\alpha}\cdot\boldsymbol{\Omega}_2(\mathbf{x}) & \ldots & 0 & -i\omega_{p,2}(\mathbf{x}) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \omega + \boldsymbol{\alpha}\cdot\boldsymbol{\Omega}_{\mathcal{N}}(\mathbf{x}) & -i\omega_{p,\mathcal{N}}(\mathbf{x}) & 0 \\ i\omega_{p,1}(\mathbf{x}) & i\omega_{p,2}(\mathbf{x}) & \ldots & i\omega_{p,\mathcal{N}}(\mathbf{x}) & \omega & -ic\boldsymbol{\alpha}\cdot\mathbf{k} \\ 0 & 0 & \ldots & 0 & ic\boldsymbol{\alpha}\cdot\mathbf{k} & \omega \end{pmatrix}}$$

Weyl expansion of a dispersion operator

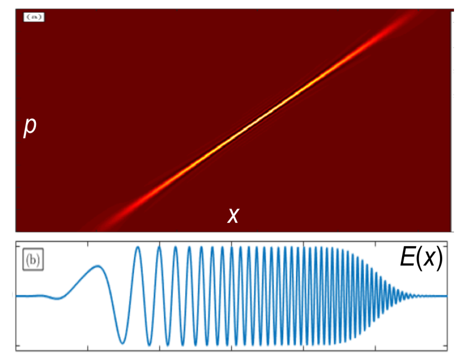
- For an oscillating field $\psi = \mathrm{Re}\,(e^{i\theta}\boldsymbol{\Psi})$, one can introduce an envelope operator:

$$\hat{\mathbf{D}}\psi = 0 \quad \Rightarrow \quad \hat{\boldsymbol{\mathcal{D}}}\boldsymbol{\Psi} = 0, \quad \hat{\boldsymbol{\mathcal{D}}} = e^{-i\theta(\mathbf{x})}\hat{\mathbf{D}}\,e^{i\theta(\mathbf{x})}$$

- $\hat{\boldsymbol{\mathcal{D}}} = \hat{\boldsymbol{\mathcal{D}}}(\mathbf{x}, -i\partial_{\mathbf{x}})$ can be mapped to a function, approximated using $\epsilon = \lambda/L$ as a small parameter, and then mapped backed to the operator space.

- A suitable mapping is the Wigner-Weyl transform:

$$\hat{\boldsymbol{W}}_{\boldsymbol{E}} = \boldsymbol{E}\boldsymbol{E}^{\dagger} \mapsto \boldsymbol{W}_{\boldsymbol{E}}$$



$$\mathfrak{W} : \hat{A} \mapsto A(\mathbf{x}, \mathbf{p}) = \int \langle \mathbf{x} + \mathbf{s}/2 | \hat{A} | \mathbf{x} - \mathbf{s}/2 \rangle\, e^{-i\mathbf{p}\cdot\mathbf{s}}\, d^4\mathbf{s}$$

$$\mathfrak{W}^{-1} : A \mapsto \hat{A} = \int |\mathbf{x} - \mathbf{s}/2\rangle\, A(\mathbf{x}, \mathbf{p})\, e^{-i\mathbf{p}\cdot\mathbf{x}}\, \langle \mathbf{x} + \mathbf{s}/2|\; d^4\mathbf{x}\, d^4\mathbf{p}\, \frac{d^4\mathbf{s}}{(2\pi)^4}$$

$$f(\hat{\mathbf{x}}) \leftrightarrow f(\mathbf{x}), \quad f(\hat{\mathbf{p}}) \leftrightarrow f(\mathbf{p}), \quad \tfrac{1}{2}\,[f(\hat{\mathbf{x}})\hat{\mathbf{p}} + \hat{\mathbf{p}}f(\hat{\mathbf{x}})] \leftrightarrow f(\mathbf{x})\mathbf{p}$$

$$\hat{\boldsymbol{\mathcal{D}}} \approx \underbrace{\mathbf{D}(\mathbf{x}, \partial_{\mathbf{x}}\theta)}_{O(\epsilon^0)} + \underbrace{\tfrac{1}{2}\,[(-i\partial_\alpha) \circ \boldsymbol{\mathcal{V}}^\alpha + \boldsymbol{\mathcal{V}}_\alpha \circ (-i\partial^\alpha)]}_{O(\epsilon^1)} + \underbrace{\tfrac{1}{2}\,(-i\partial_\alpha) \circ \boldsymbol{\vartheta}^{\alpha\beta} \circ (-i\partial_\beta) - \tfrac{1}{8}\,(\partial^2_{\alpha\beta}\boldsymbol{\vartheta}^{\alpha\beta}) + \dots}_{O(\epsilon^2)}$$

$$\mathbf{k}(\mathbf{x}) = \partial_{\mathbf{x}}\theta(\mathbf{x}), \quad \boldsymbol{\mathcal{V}}^\alpha(\mathbf{x}) = [\partial\mathbf{D}(\mathbf{x}, \mathbf{p})/\partial p_\alpha]_{\mathbf{p}=\mathbf{k}(\mathbf{x})}, \quad \boldsymbol{\vartheta}^{\alpha\beta}(\mathbf{x}) = [\partial^2\mathbf{D}(\mathbf{x}, \mathbf{p})/\partial p_\alpha\partial p_\beta]_{\mathbf{p}=\mathbf{k}(\mathbf{x})}$$

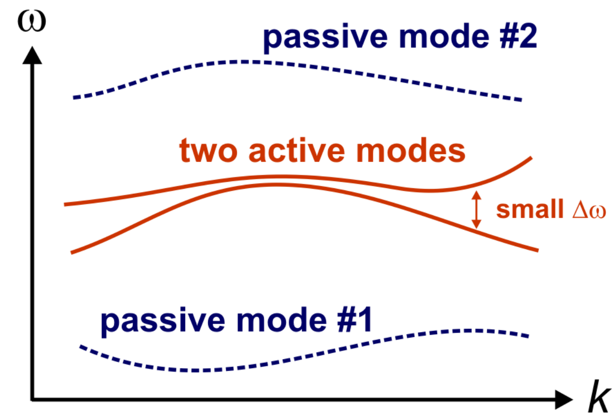
Active and passive modes

$$0 = \hat{\boldsymbol{\mathcal{D}}}\boldsymbol{\Psi} \approx [\mathbf{D}(\mathbf{x},\mathbf{k}(\mathbf{x})) + \epsilon\hat{\boldsymbol{\mathcal{D}}}^{(1)} + \epsilon^2\hat{\boldsymbol{\mathcal{D}}}^{(2)} + \ldots]\boldsymbol{\Psi}$$

- Let us represent the field in the basis of $\mathbf{D}(\mathbf{x},\mathbf{k}(\mathbf{x}))$'s eigenvectors $\boldsymbol{\eta}_q$:

$$\boxed{\boldsymbol{\Psi} = \textstyle\sum_{q=1}^{N} \boldsymbol{\eta}_q a_q + \sum_{q=N+1}^{N_c} \boldsymbol{\eta}_q \bar{a}_q}$$

$a_q = O(\epsilon^0)$ – "active" modes
$\bar{a}_q = O(\epsilon^1)$ – "passive" modes



- Active modes are excited externally or through mode-conversion. Passive modes are adiabatic.

- By projecting the wave equation on the active- and passive-mode subspaces, one can obtain a reduced equation that contains active modes only:

$$\begin{aligned} &\hat{\boldsymbol{\mathcal{D}}}_{aa}\mathbf{a} + \hat{\boldsymbol{\mathcal{D}}}_{a\bar{a}}\bar{\mathbf{a}} = 0 \\ &\hat{\boldsymbol{\mathcal{D}}}_{\bar{a}a}\mathbf{a} + \underbrace{\hat{\boldsymbol{\mathcal{D}}}_{\bar{a}\bar{a}}}_{O(1)}\bar{\mathbf{a}} = 0 \end{aligned} \quad \Rightarrow \quad \bar{\mathbf{a}} = -\hat{\boldsymbol{\mathcal{D}}}_{\bar{a}\bar{a}}^{-1}\hat{\boldsymbol{\mathcal{D}}}_{\bar{a}a}\mathbf{a} \quad \Rightarrow \quad \boxed{[\underbrace{\hat{\boldsymbol{\mathcal{D}}}_{aa}}_{O(\epsilon)} - \underbrace{\hat{\boldsymbol{\mathcal{D}}}_{a\bar{a}}}_{O(\epsilon)}\underbrace{\hat{\boldsymbol{\mathcal{D}}}_{\bar{a}\bar{a}}^{-1}}_{O(1)}\underbrace{\hat{\boldsymbol{\mathcal{D}}}_{\bar{a}a}}_{O(\epsilon)}]\mathbf{a} \approx 0}$$

The contribution of passive modes is $O(\epsilon^2)$, so it cannot be neglected.

The resulting equation for active modes

$$\boxed{\Big[\underbrace{\boldsymbol{\mathfrak{D}}}_{O(1)}\underbrace{-i\mathbf{V}\cdot\nabla-\tfrac{i}{2}(\nabla\cdot\mathbf{V})+i\boldsymbol{\Gamma}-\mathbf{U}}_{O(\epsilon)}\underbrace{+\boldsymbol{\Xi}^\dagger\hat{\boldsymbol{\Delta}}\boldsymbol{\Xi}-\boldsymbol{\Xi}^\dagger\hat{\boldsymbol{\Lambda}}\bar{\boldsymbol{\Xi}}\bar{\boldsymbol{\mathfrak{D}}}^{-1}\bar{\boldsymbol{\Xi}}^\dagger\hat{\boldsymbol{\Lambda}}\boldsymbol{\Xi}}_{O(\epsilon^2)}\Big]\mathbf{a}=0}$$

$$\boldsymbol{\mathfrak{D}}=\boldsymbol{\Xi}^\dagger\mathbf{D}_H\boldsymbol{\Xi}=\mathrm{diag}\{\lambda_1\ldots\lambda_N\},\quad \boldsymbol{\Xi}=(\boldsymbol{\eta}_1\ \ldots\ \boldsymbol{\eta}_N)$$

$$\bar{\boldsymbol{\mathfrak{D}}}=\bar{\boldsymbol{\Xi}}^\dagger\mathbf{D}_H\bar{\boldsymbol{\Xi}}=\mathrm{diag}\{\lambda_{N+1}\ldots\lambda_{N_c}\},\quad \bar{\boldsymbol{\Xi}}=(\boldsymbol{\eta}_{N+1}\ \ldots\ \boldsymbol{\eta}_{N_c})$$

$$\mathbf{V}=\partial_{\mathbf{k}}\boldsymbol{\mathfrak{D}}+\tilde{\mathbf{V}},\quad \tilde{\mathbf{V}}=i\boldsymbol{\Xi}^\dagger(\partial_{\mathbf{k}}\mathbf{D}_A)\boldsymbol{\Xi}-(\partial_{\mathbf{k}}\boldsymbol{\Xi}^\dagger)\mathbf{D}_H\boldsymbol{\Xi}-\boldsymbol{\Xi}^\dagger\mathbf{D}_H(\partial_{\mathbf{k}}\boldsymbol{\Xi})=O(\epsilon)$$

$$\boldsymbol{\Gamma}=\boldsymbol{\Xi}^\dagger\mathbf{D}_A\boldsymbol{\Xi},\quad \mathbf{U}=\tfrac{i}{2}\big[\boldsymbol{\Xi}^\dagger\boldsymbol{\mathcal{V}}^\alpha(\partial_\alpha\boldsymbol{\Xi})-(\partial_\alpha\boldsymbol{\Xi}^\dagger)\boldsymbol{\mathcal{V}}^\alpha\boldsymbol{\Xi}\big]$$

$$\hat{\boldsymbol{\Delta}}=\tfrac{1}{2}(-i\partial_\alpha)\circ\boldsymbol{\vartheta}^{\alpha\beta}(\mathbf{x},\mathbf{k}(\mathbf{x}))\circ(-i\partial_\beta)-\tfrac{1}{8}\partial^2_{\alpha\beta}\vartheta^{\alpha\beta}(\mathbf{x},\mathbf{k}(\mathbf{x}))$$

$$\hat{\boldsymbol{\Lambda}}=\tfrac{1}{2}\left[(-i\partial_\alpha)\circ\boldsymbol{\mathcal{V}}^\alpha(\mathbf{x},\mathbf{k}(\mathbf{x}))+\boldsymbol{\mathcal{V}}^\alpha(\mathbf{x},\mathbf{k}(\mathbf{x}))\circ(-i\partial_\alpha)\right]+i\mathbf{D}_A(\mathbf{x},\mathbf{k}(\mathbf{x}))$$

- **Method:** (i) use ray tracing with Hamiltonian $\mathrm{Tr}\,\boldsymbol{\mathfrak{D}}(\mathbf{x},\mathbf{k})$ to find $\mathbf{k}(\mathbf{x})$ that eliminates $O(1)$; (ii) calculate the remaining matrices and solve for $\mathbf{a}(\mathbf{x})$.
- The density of power absorption can be calculated using $\boldsymbol{P}_{\mathrm{abs}}\approx(\boldsymbol{\omega}/8\boldsymbol{\pi})\,\mathbf{a}^\dagger\boldsymbol{\Gamma}\mathbf{a}$.

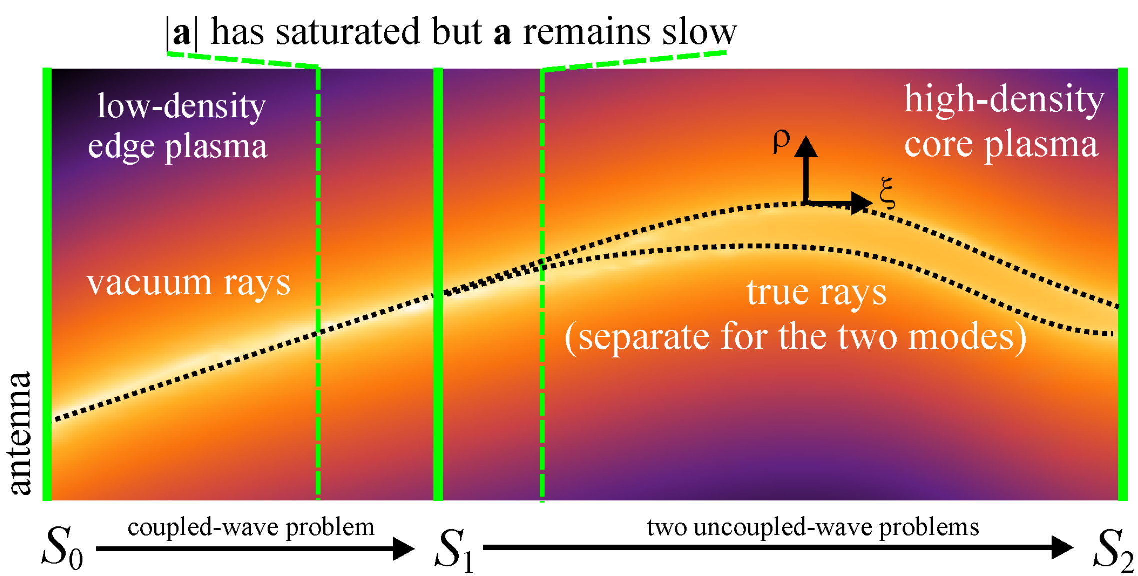
Quasioptical model for EM beams in weakly inhomogeneous media

- Consider ray-based coordinates with ξ along rays and $\boldsymbol{\rho}$ perpendicular to rays. If $\partial_\xi \ll \partial_\rho$, then $O(\partial_\xi^2)$ is negligible. Denote $\bar{\mathcal{E}} = N^{-1}\mathrm{Tr}\,\boldsymbol{\mathfrak{D}}$ and $\widetilde{\boldsymbol{\mathcal{E}}} = \boldsymbol{\mathfrak{D}} - \bar{\mathcal{E}}\mathbb{1}$. Then,

$$\boxed{\left[\mathbb{1}\,\frac{\partial \bar{\mathcal{E}}}{\partial k_\xi}\,\frac{\partial}{\partial \boldsymbol{\xi}} + \delta \mathbf{V}^\mu \frac{\partial}{\partial \rho^\mu} - \frac{i}{2}\,\frac{\partial}{\partial \boldsymbol{\rho}^{\boldsymbol{\mu}}}\,\boldsymbol{\Phi}^{\mu\nu}\,\frac{\partial}{\partial \boldsymbol{\rho}^{\boldsymbol{\nu}}} + \frac{1}{2}(\nabla \cdot \mathbf{V}) - \boldsymbol{\Gamma} + i(\widetilde{\boldsymbol{\mathcal{E}}} - \mathrm{U})\right]\mathbf{a} = 0}$$

$$\delta \mathbf{V}^\mu = \left(\widetilde{\mathbf{V}}^\alpha + \frac{\partial \widetilde{\boldsymbol{\mathcal{E}}}}{\partial k_\alpha}\right)\frac{\partial \rho^\mu}{\partial x^\alpha}, \qquad \boldsymbol{\Phi}^{\mu\nu} = \frac{\partial \rho^\mu}{\partial x^\alpha}\,\boldsymbol{\Xi}^\dagger\left(\boldsymbol{\vartheta}^{\alpha\beta} - 2\boldsymbol{\mathcal{V}}^\alpha \bar{\boldsymbol{\Xi}}\bar{\boldsymbol{\mathfrak{D}}}^{-1}\bar{\boldsymbol{\Xi}}^\dagger \boldsymbol{\mathcal{V}}^\beta\right)\boldsymbol{\Xi}\,\frac{\partial \rho^\nu}{\partial x^\beta}$$

- Proposed method for modeling O-X coupling on LHD:



Reduced analytical theory of the O-X mode conversion at the edge

- If, at the edge, one can adopt constant $\mathbf{k} = (\omega/c)\mathbf{e}_k$ and neglect both diffraction and dissipation, then the O-X mode conversion is tractable analytically.

$$\big[\mathfrak{D} - i\mathbf{V}\cdot\nabla - (i/2)\,(\nabla\cdot\mathbf{V}) - \mathbf{U}\big]\mathbf{a} \approx 0$$

  - Assuming the wave equation in the form $\hat{\mathbf{D}}\mathbf{E} = 0$, one can use

$$\mathbf{D}(\mathbf{x},\mathbf{p}) = \underbrace{(c/\omega)^2(\mathbf{p}\mathbf{p}^\dagger - p^2\mathbb{1}_3) + \mathbb{1}_3}_{\mathbf{D}_0(\mathbf{p})} + \underbrace{\boldsymbol{\chi}(\mathbf{x})}_{\text{perturbation}}$$

  - Approximate eigenvalues of $\mathbf{D}$ (cf. the quantum formula $\Delta E_q \approx \langle\psi_q|H_1|\psi_q\rangle$):

$$D_q \approx D_0(\mathbf{p}) + (\boldsymbol{\eta}_q^\dagger\boldsymbol{\chi}\boldsymbol{\eta}_q)(\mathbf{x},\mathbf{p}), \quad D_0 = 1 - (pc/\omega)^2, \quad \mathbf{V} \approx (-2\mathbf{k}/k^2)\mathbb{1}$$

$$\mathfrak{D} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad \mathbf{U} = -2i\begin{pmatrix} \boldsymbol{\eta}_1^\dagger\cdot\boldsymbol{\eta}_1' & \boldsymbol{\eta}_1^\dagger\cdot\boldsymbol{\eta}_2' \\ -(\boldsymbol{\eta}_1^\dagger\cdot\boldsymbol{\eta}_2')^* & \boldsymbol{\eta}_2^\dagger\cdot\boldsymbol{\eta}_2' \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Dodin, Ruiz, and Kubo, arXiv:1709.02841.

The equation for the coupled-mode amplitudes on a ray

- Polarization in the vacuum limit $[\vartheta = \angle(\mathbf{k}, \mathbf{B}_0),\ g_{1,2} = u^{-1} \mp (\operatorname{sgn} u)\sqrt{1+u^{-2}}]$:

$$\boldsymbol{\eta}_1 = \frac{1}{\sqrt{1+g_1^2}} \begin{pmatrix} -\cos\vartheta \\ i g_1 \\ \sin\vartheta \end{pmatrix}, \qquad \boldsymbol{\eta}_2 = -\frac{i \operatorname{sgn} u}{\sqrt{1+g_2^2}} \begin{pmatrix} -\cos\vartheta \\ i g_2 \\ \sin\vartheta \end{pmatrix}$$

- The equation for $\mathbf{a}$ is similar to that governing a two-level quantum system:

$$\boxed{i\mathbf{a}' = \boldsymbol{\mathcal{H}}\mathbf{a}}$$

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad \boldsymbol{\mathcal{H}} = \begin{pmatrix} \epsilon_3 - \rho/2 & \epsilon_1 - i\epsilon_2 \\ \epsilon_1 + i\epsilon_2 & -\epsilon_3 + \rho/2 \end{pmatrix} \equiv \begin{pmatrix} -\alpha & -i\beta \\ i\beta^* & \alpha \end{pmatrix}$$

$$\epsilon_1 = -\frac{u'}{2(1+u^2)} = \frac{\delta'}{2}, \quad \epsilon_2 = \frac{\zeta}{\sqrt{1+u^2}} = \zeta\cos\delta, \quad \epsilon_3 = -\frac{\zeta u}{\sqrt{1+u^2}} = \zeta\sin\delta$$

$$u = \frac{2\omega(\mathbf{e}_k \cdot \mathbf{b})}{\Omega(\mathbf{e}_k \times \mathbf{b})^2}, \qquad \zeta = \frac{(\mathbf{e}_k \times \mathbf{b}) \cdot \mathbf{b}'}{(\mathbf{e}_k \times \mathbf{b})^2}, \qquad \rho = \frac{2|\Omega|\omega_p^2}{\omega(\omega^2 - \Omega^2)} \sqrt{(\mathbf{e}_k \cdot \mathbf{b})^2 + \frac{\Omega^2}{4\omega^2}(\mathbf{b} \times \mathbf{e}_k)^4}$$

Dodin, Ruiz, and Kubo, arXiv:1709.02841.

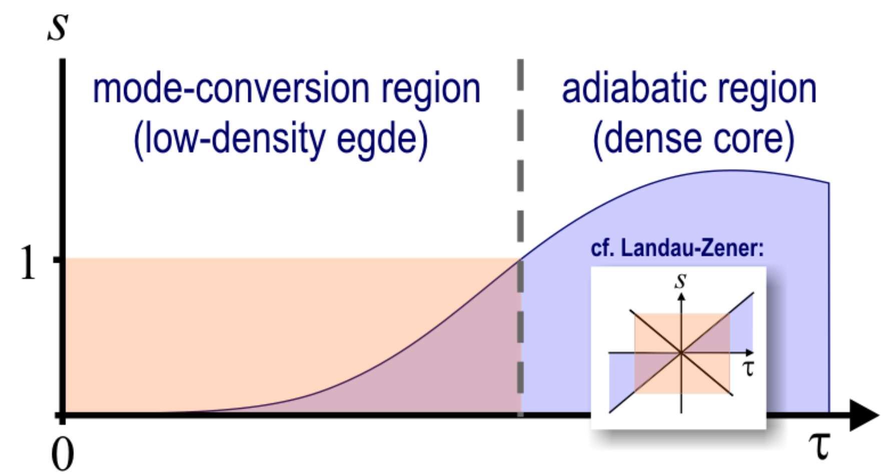
Mode conversion in dense plasma is *special*.

- In the right variables, $i\mathbf{a}' = \boldsymbol{\mathcal{H}}\mathbf{a}$ is parametrized **by just one real function, s**:

$$q_{1,2} = a_{1,2}e^{\mp i\gamma/2}, \quad \gamma = \arg\beta, \quad s = (\alpha/|\beta|) - \dot{\gamma}/2, \quad \tau = \int |\beta|\, dl$$

$$\boxed{i\,\frac{d}{d\tau}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} -s & -i \\ i & s \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}}$$

$$\ddot{q}_1 + \left(1 + s^2 - i\dot{s}\right) q_1 = 0$$



- Similar equations emerge also in other mode-conversion theories, but in those works, $s(\tau)$ is usually approximated by a linear function under the assumption that the resonance region is localized (Landau–Zener paradigm) $\Rightarrow$ Weber equation.

Mode conversion in edge plasma is qualitatively different from the standard Landau-Zener problem because, at the edge, s is strongly nonlinear.

Dodin, Ruiz, and Kubo, arXiv:1709.02841; cf. Tracy *et al*, *Ray Tracing and Beyond* (2014).

The amplitude dynamics is understood using a quantum analogy.

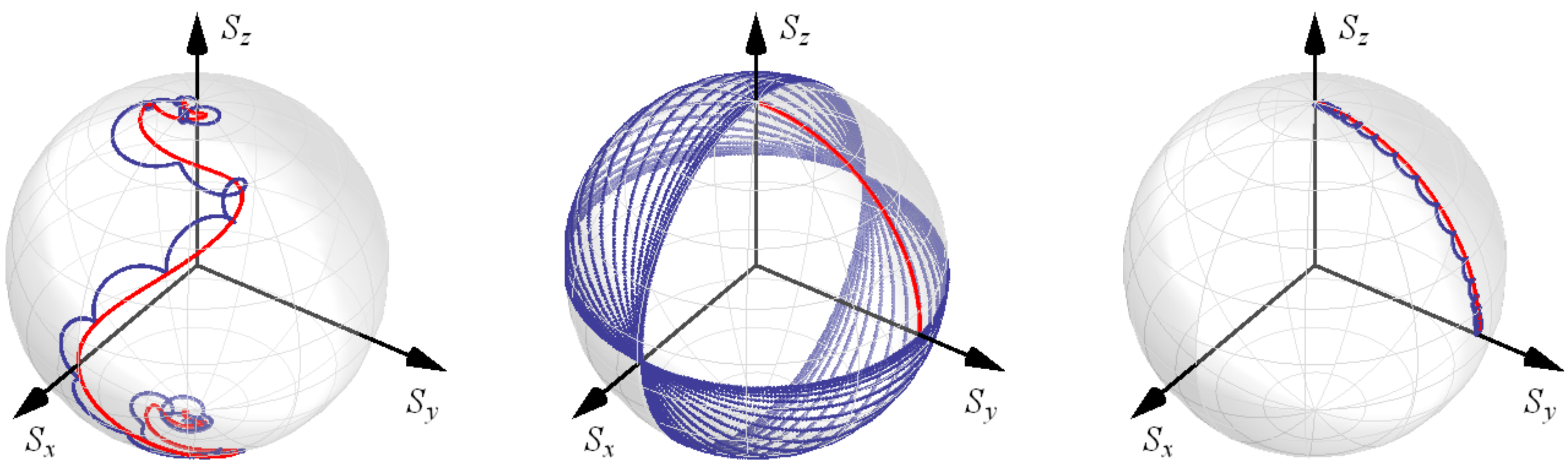
- The 2D equation for complex $\mathbf{a}$ can be written as a 3D equation for a real "spin":

$$\boldsymbol{\mathcal{H}} = \frac{1}{2}\boldsymbol{\sigma}_\mu \mathcal{B}^\mu, \quad \boldsymbol{\mathcal{B}} = (\epsilon_1, \epsilon_2, \epsilon_3 - \rho/2)^{\mathrm{T}}, \quad S_\mu = \mathbf{a}^\dagger \boldsymbol{\sigma}_\mu \mathbf{a}$$

$$\boxed{i\mathbf{a}' = \boldsymbol{\mathcal{H}}\mathbf{a} \quad \Rightarrow \quad \mathbf{S}' = \boldsymbol{\mathcal{B}} \times \mathbf{S}}$$

- If $\boldsymbol{\mathcal{B}}$ is slow, the precession plane remains normal to $\boldsymbol{\mathcal{B}}$. In dense plasma, $\boldsymbol{\mathcal{B}} \to \mathbf{e}_z \mathcal{B}_z$, so having a pure mode $(\mathbf{S} \parallel \mathbf{e}_z)$ implies $\mathbf{S} \parallel \boldsymbol{\mathcal{B}}$. **This is ensured by using $\mathbf{S}_0 \parallel \boldsymbol{\mathcal{B}}_0$.**

- For $\vartheta = \pi/2$, this means $\mathbf{S}_0 \parallel \mathbf{e}_y$, so *circular polarization* is needed in vacuum.



Dodin, Ruiz, and Kubo, arXiv:1709.02841.

A more formal WKB analysis leads to the same results.

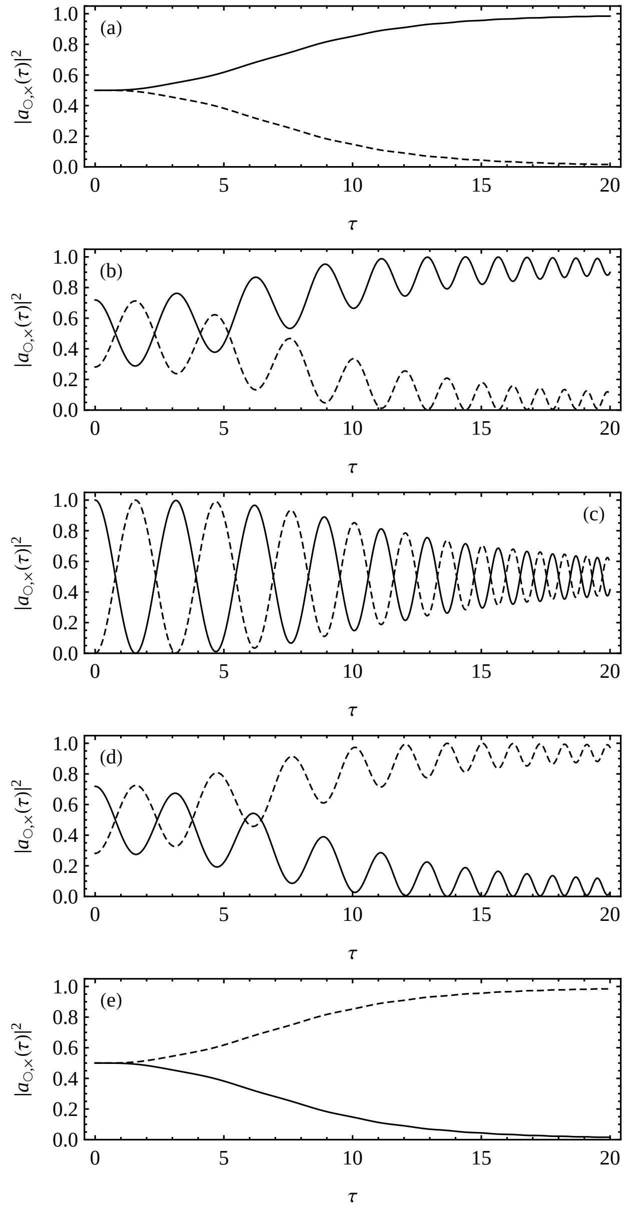
$$\ddot{q}_1 + \underbrace{(1 + s^2 - i\dot{s})}_{Q} q_1 = 0$$

$$\frac{d}{d\tau}\left(\frac{2\pi}{\sqrt{Q}}\right) \ll 1, \quad \text{i.e.,} \quad \dot{s} \ll 1$$

$$a_1(\tau) \approx \frac{1}{[Q(\tau)]^{1/4}}\left[C_+ e^{i\phi(\tau)} + C_-\, e^{-i\phi(\tau)}\right]$$

$$\operatorname{Re}\phi \approx \pm \int_0^{\tau} \sqrt{1 + s^2(\tilde{\tau})}\, d\tilde{\tau}$$

$$\operatorname{Im}\phi = -\frac{1}{2}\ln\left(|s| + \sqrt{1 + s^2}\right)$$



Dodin, Ruiz, and Kubo, arXiv:1709.02841.

Summary

- Background [Daniel Ruiz's PhD thesis (2017)]:
  - GO corresponds to the 0th-order expansion of the action in $\epsilon = \lambda/L$.
  - Extended geometrical optics (XGO) is the 1st-order theory. It captures mode conversion and the polarization-driven modification of the ray trajectories.

- New applications/extensions of XGO motivated by a collaboration with NIFS:
  - The **second-order theory** (XXGO) has been developed that captures diffraction.
  - Based on that, a **quasioptical model** of wave-beam propagation has been proposed for weakly inhomogeneous media. A numerical implementation is now being developed for modeling ECH on LHD at NIFS.
  - In parallel, the problem of **O-X mode conversion at the plasma edge** has been solved analytically. It does not fit into the standard Landau-Zener paradigm. Nevertheless, it has been made tractable by using the analogy between the coupled-mode dynamics and the spin precession of a spin-$1/2$ particle.

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